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The Thermodynamic Approach to Fluidized Drying and Moistening Optimization

For drying and moistening processes in batch fluidization and fluidization in a horizontal exchanger, the overall mathematical model describing simultaneous heat and mass exchange is given. On this basis, the problem of finding the optimal temperatures of inlet gas is formulated and investigated for the case when inlet gas humidity is constant. It is explained that when unit economical values of gas and solid phases are linear with regard to unit available energies (exergies), the economical problem of minimum costs is equivalent to the thermodynamical problem of minimum available energy (exergy) dissipation. Therefore, the performance index expression in exergy terms is formulated as a functional of the Bolza form. The dynamic programming method is used to obtain results of optimal trajectories and decisions. The dependence of optimal process time on apparatus price, which is characterized here by a Lagrangian multiplier λ, is pointed out.

The results indicate that the constant gas inlet temperature policy commonly applied to the processes considered here prevents attainment of the process optimum. In fact, increasing inlet temperature until an admissible gas temperature is attained is the optimal policy.

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SCOPE

To the class of important processes between gases and granular solids, where the use of variational methods is required for optimization, belong the process of batch fluidization and steady state fluidization process in the horizontal exchanger (Figure 1). In the first process, the solid state changes as a function of time; in the second, it changes as a function of distance when ideal mixing in the first apparatus and ideal mixing in vertical cross-section of the second apparatus is assumed. In an isobaric and adiabatic process, the thermodynamic state of the solid is described by enthalpy I_s and moisture content W_s, the gas by enthalpy i_g and humidity X_g.

In this article the optimal transition of solid phase from one thermodynamic state I_{so}, W_{so} to another I_{sk}, W_{sk} is considered for the drying or moistening processes shown on Figure 1. This transition occurs during an isobaric and adiabatic process due to simultaneous energy and mass exchange between gas and solid. The total time of such transition is finite.

The usual way of conducting the processes considered is with a constant inlet gas state. In such a case the inlet gas enthalpy i_g and absolute humidity X_g do not change with solid residence time in either of the processes. It is known that driving forces of such processes are large at the beginning and small at the end of the process and that this fact makes the process uneconomical, that is, causes large production costs. Therefore, it should be expected that variable inlet gas states should make the processes considered more economical. The finding of a technique which minimizes production costs is a task of optimization.

It will be shown that in both of the processes considered,

the heat and mass exchange are described by the same mathematical model. Therefore, it is useful to consider the optimization problem for both processes simultaneously. The solid enthalpy I_s and solid moisture content W_s are state variables in the original formulation of optimization problem. The inlet gas enthalpy i_g and the inlet gas humidity X_g *are decision* variables in the most general case. In this paper, the optimization is considered just for the case where the enthalpy (or temperature) of the inlet gas is the only decision variable with the required gas humidity X_g constant and equal to X_{go}. This case is very common because attaining the necessary change of the gas humidities in time is quite cumbersome. The problem of the optimization of the fluidization process for the case where temperature and humidity of the inlet gas are changing is mathematically more difficult and will be the theme of a separate publication.

The sets of equations describing the energy and mass exchange in the batch process as well as the process in a horizontal fluidized exchanger are derived for the general case when the Lewis number differs from one, and the outlet gas is not in equilibrium with the solid (finite value of mass transfer units number N_g). After that, the solution of the optimization problem in the case of equilibrium between outlet gas and solid ($N_g \rightarrow \infty$) is discussed. This case occurs commonly in real fluidizing apparata where the height of the bed is of order 0.5 m, due to the very large specific surface of the solid and consequent very small height of the transfer units HTU.

In view of the uncertainty of gas and solid unit values as a function of their thermodynamic states, attention is directed toward providing an optimization solution which is independent of price fluctuations, the location of the process, and the date of process performance. In order to accomplish this, the unit economic values of solid and gas are assumed as linear with regard to the unit available energies (exergies).

As it is briefly explained in this paper and as it is known from the work by Evans and Tribus (1965), the available energy (not total energy) is a thermodynamic measure of substance value. Therefore, the unit economical values are assumed as linear with regard to the unit available energies (exergies). This leads finally to the expression of a process performance index in exergy terms. The investigation of exergy performance indexes is a task of thermodynamic optimization.

CONCLUSIONS AND SIGNIFICANCE

In the case when the unit economical values of gas and solid are linear with regard to the respective unit available energies (exergies), the thermodynamic optimization gives the same results as conventional minimization of production costs. If the assumption about linearity cannot be made,* then the results of economic and thermodynamic optimization differ. However, in order to obtain the optimal results in this case, it suffices to replace unit exergies in the expression for performance index by any other unit measures of substance value which are more appropriate for the specific investigation considered, for example, unit economic prices. It should be pointed out that such replacement does not change the computational procedure.

In the case when optimization based on unit exergies and optimization based on unit economic prices are not equivalent, the results of the economical optimization are more useful in the specific current applications. However, the thermodynamic optimization gives more universal results whose relevance is independent of economic data uncertainties, date of process performance, place, current economical policy, and any nonphysical influences on the process. Therefore, results obtained by thermodynamic approach have more future importance. Since they provide the solution which is independent of time and place we obtain the standard of the comparison of the real process which is equally valuable in current and future investigations. For this reason, further development of thermodynamic ideas should be expected in optimization.

In the pure thermodynamic analysis, the constraint on *the total* process time is most often neglected (Bosniakovic, 1960). Such an analysis only leads to the trivial conclusion that the optimal process is a quasistatic one. Since a quasistatic process has a rate approaching zero, the practical value of this conclusion is very low.

Because the mathematical model of thermodynamic optimization pertains to processes having a definite duration, a much more general family of optimal processes is obtained in the considered approach. This family is comprised of real, irreversible processes. The present analysis is the first such approach in the engineering literature that examines systematically the relation between optimal process intensity, optimal time, and apparatus price (represented here as the Lagrangian multiplier λ).

On the basis of the results of this work, we can draw the conclusion that the reversible processes will be optimal only in the case when the apparatus price approaches zero ($\lambda = 0$). The optimal process time then approaches infinity. In general, when the apparatus price is greater than zero ($\lambda > 0$), only irreversible processes will be optimal. The optimal irreversible processes are carried out with the finite driving forces $i_g - i_s$ and $X_g - X_s$ which increase with the numerical value of the parameter λ. When a very expensive apparatus is required, then the optimal processes must be carried out at a high intensity in order to assure short process time. The above results help to avoid basic mistakes in process design and apparata design.

The specific results of thermodynamic optimization are presented here for the drying process in batch fluidization and fluidization in a horizontal exchanger. They indicate that optimal gas temperature policy corresponds with an inlet gas temperature that increases with time until a constant admissible gas temperature is attained. This means that the usual method of conducting the processes considered here, based on the technique of the constant inlet gas state, prevents the attainment of the optimum of the process. Therefore, an increasing temperature policy should be recommended in the considered fluidized drying operations. The design of fluidization processes with the variable inlet gas conditions should be made with the help of the results of optimization (see example). The practical attainment of a variable gas temperature policy is especially simple in the case of batch fluidization. The time-controlled heating of gas supplied to the fluidized apparatus is sufficient for the realization of the optimal policy. Such a policy makes the cost of production of order 40 ÷ 20% less than costs found in the case of the constant temperature.

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* This can occur, for example, in the case when a large exergy density is especially demanded. This density represents then an additional parameter influencing the unit economic value of a substance. (Szargut and Petela, 1965).

THE PROCESS EQUATIONS

The conservation and kinetic equations for the batch fluidization (Figure 1, scheme A) are formulated with the assumption that gas state changes in the apparatus are neglected. This assumption is allowed because of the small void fraction in the bed. Also, the common assumption about ideal mixing of gas and solid phases is accepted. Taking into account the above assumptions, one can describe the batch fluidization process by a set of ordinary differential equations which describe the changes of solid enthalpy I_s and solid moisture content W_s in time. At first, the balance equations are obtained. These are as follows:

$$S_o \frac{dI_s}{d\eta} = G(i_g - i_w) \quad \text{energy balance equation} \qquad (1)$$

$$S_o \frac{dW_s}{d\eta} = G(X_g - X_w) \quad \text{mass balance equation} \qquad (2)$$

The left-hand side of the above equations describes energy and mass accumulation in the bed per unit time, respectively. The right-hand side of the equations describes a difference between the streams of the inlet and outlet energy and mass, respectively. According to the laws of conservation, the resulting streams are equal to the accumulation of energy and mass in the bed per unit time.

On the other hand, the changes of solid enthalpy and solid moisture content in the unit time should be obtained from the kinetic considerations. Because ideal mixing occurs, the overall driving forces can operate with differences between outlet gas potentials and potentials of gas in equilibrium with solid. Using the mass transfer driving force expressed as the difference between outlet gas stream humidity X_w and humidity of gas at equilibrium with solid X_s, one can write the kinetic equations in the form

$$S_o \frac{dI_s}{d\eta} \doteq \alpha\, a(t_w - t_s)V + i_p K_g'\, a(X_w - X_s)V \qquad (3)$$

energy exchange equation

$$S_o \frac{dW_s}{d\eta} = K_g'\, a(X_w - X_s)V \qquad (4)$$

mass exchange equation

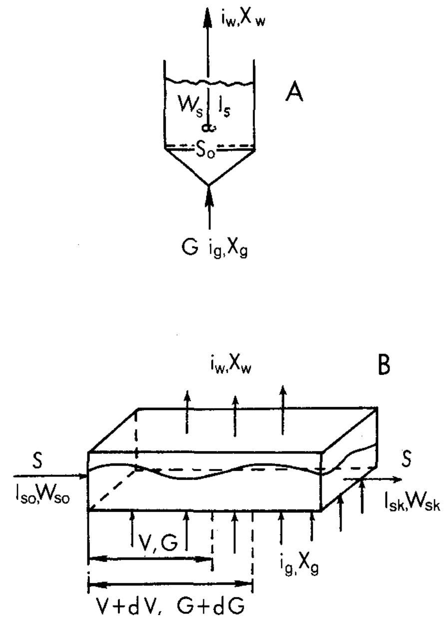

Fig. 1. Scheme of the batch fluidization (A) and fluidization in the horizontal exchanger (B).

The right-hand side of the energy transfer equation, Equation (3), consists of two terms. The first describes the stream of sensible heat, the second, the energy transferred with the mass of moisture.

Since the enthalpies and concentrations of both phases are used in the process description, the right-hand side of Equation (3) should be expressed as a function of gas enthalpy and gas humidity. For this purpose we can use the following approximate equation:

$$i_w - i_s = \overline{C}_g(t_w - t_s) + i_p(X_w - X_s) \qquad (5)$$

in Equation (3). Transforming the set obtained to dimensionless form yields

$$\frac{dI_s}{d\tau} = i_g - i_w \qquad \text{energy balance equation} \qquad (6)$$

$$\frac{dW_s}{d\tau} = X_g - X_w \qquad \text{mass balance equation} \qquad (7)$$

$$\frac{dI_s}{d\tau} = \frac{N_g}{Le}\{(i_w - i_s) + i_p(Le - 1)(X_w - X_s)\}$$

$$\text{energy transfer equation} \qquad (8)$$

$$\frac{dW_s}{d\tau} = N_g(X_w - X_s) \quad \text{mass transfer equation} \qquad (9)$$

We ask the reader to prove that above dimensionless set of equations is also valid for the fluidization in a horizontal exchanger (Figure 1, scheme B) if time τ is appropriately defined (see Notation) and if perfect mixing in vertical plane is assumed.

It is reasonable to point out that the set of Equations (6) through (9) does not represent the state equations which are usually found in optimization because these equations contain not only the state variables I_s, W_s and decisions i_g, X_g but also additional variables i_w, X_w which describe the outlet gas state. Thus, it is necessary to eliminate variables i_w, X_w from Equations (6) through (9) in order to break down the discussed set into the typical state equations form. Equating the right-hand sides of Equations (6) and (8) and also (7) and (9), and determining from the resulting equations the enthalpy i_w and the humidity X_w gives

$$i_w = \left(i_g + \frac{N_g i_s}{Le} - \frac{N_g i_p (Le - 1)(X_g - X_s)}{Le(N_g + 1)}\right) \Big/ \left(\frac{N_g}{Le} + 1\right) \qquad (10)$$

$$X_w = (X_g + N_g X_s)/(N_g + 1) \qquad (11)$$

Substituting Equation (10) into Equation (6) and Equation (11) into Equation (7) yields

$$\frac{dI_s}{d\tau} = \frac{N_g}{Le}\left(i_g - i_s[I_s, W_s] + \frac{i_p(Le - 1)(X_g - X_s[I_s, W_s])}{N_g + 1}\right) \Big/ \left(\frac{N_g}{Le} + 1\right) \qquad (12)$$

$$\frac{dW_s}{d\tau} = \frac{N_g(X_g - X_s[I_s, W_s])}{N_g + 1} \qquad (13)$$

The above set of equations represents the equations of state according to terminology used in optimization. These equations have on their right-hand sides only the decision variables (i_g, X_g), thermodynamic functions of state variables $i_s\,[I_s, W_s]$, $X_s\,[I_s, W_s]$, and constant process parameters Le, N_g.

THE PROCESS PERFORMANCE INDEX

In order to formulate the expression for the performance index we will start with the economic considerations. Because the process is carried out in a definite apparatus (constant investment cost), it is reasonable to accept the cost of production as an optimization criterion. The production cost of the batch fluidization process is equal to the difference between the total input value of substances delivered to the system and the total output value of substances leaving it increased by the difference between the value of the substances in the system before and after the process.

For the fluidization in the horizontal exchanger a corresponding criterion involves the difference between the streams of value of all substances delivered to the system and leaving it. The performance index accepted as a production cost related to the mass unit of dry solid can be written in the common form as

$$K_e{}^p = \int_0^{\tau_k} (c_g - c_w)\,d\tau + C_{so} - C_{sk} \tag{14}$$

In order to accomplish the optimization procedure, the relationships between the unit values and the corresponding thermodynamic parameters of each phase should be known. In view of the uncertainty of gas and solid unit economic values as functions of their thermodynamic states occurring due to local fluctuation and fluctuations in time of these values, attention is directed toward providing that solution of our optimization problem which is independent of these fluctuations. It is known from thermodynamic and economic considerations that unit values of chemical substances are often accepted as proportional to unit available energies (Szargut and Petela, 1965). Such an assumption results from available energy properties, and it is admissible when the state of any merchandise product can be sufficiently described with only a set of thermodynamic parameters. It is easy to see this is so in the case here considered.

THE BASIC PROPERTIES OF AVAILABLE ENERGY (EXERGY) FUNCTION

The unit gas and solid available energies are described, respectively, by formulas

$$b_g = i_g - i_g{}^o - T^o(s_g - s_g{}^o) - v_g{}^o(p_g - P^o) - \mu^o(X_g - X_g{}^o) \tag{15}$$

$$B_s = I_s - I_s{}^o - T^o(S_s - S_s{}^o) - V_s{}^o(P_s - P^o) - \mu^o(W_s - W_s{}^o) \tag{16}$$

The available energy function is also termed exergy. This term is used mainly in European literature, but it sometimes also appears in American papers, (see, for example, Evans and Tribus, 1965). Available energy (exergy) is defined as the maximum work connected with reversible transition of a substance from its thermodynamic state to the state of equilibrium with the environment. During this process heat and mass can be exchanged between a substance and the environment. The presence of environmental parameters in the exergy definition makes exergy a quantity which is introduced on an economic basis. As a consequence of its definition and the second law of thermodynamics, exergy is a semipositive quantity. It equals zero in the state of thermodynamic equilibrium between a substance and the environment, and it is always positive outside of this equilibrium. Thus, exergy has properties of price and plays the role of a thermodynamic measure of substance value.

Exergy provides a natural and convenient means of establishing a scale of economic values of chemical substances. Since the state of these substances is most often described by thermodynamic parameters only, the economic value of any chemical substance being a state function must be related to some thermodynamic function. In the past years, the unit economic values of heating mediums (for example, values of heating steam) were established on the basis of enthalpy. However, that procedure very often gave invalid results, especially when economic values of very cold mediums were considered. The enthalpy of any substance at low temperatures has a low value. This contradicted the fact that economical values of very cold mediums are high. The exergy approach allows us to avoid this difficulty since the exergy of any cold medium has a high positive value. It turns out that the exergy approach can satisfactorily explain all weaknesses of the enthalpy approach. For more detailed discussion of this subject, the reader is referred to the book by Szargut and Petela (1965) on exergy. Several interesting remarks can also be found in the paper of Evans and Tribus (1965).

THE PERFORMANCE INDEX AS EXPRESSED IN EXERGY TERMS

The establishment of the economic values scale on the basis of exergy leads to the assumption that the unit economical values of gas and solid are linear with regard to the respective unit exergies. The slope coefficient relating the unit values with the unit exergies is here designated as equal to l. Then we have the relations:

$$c_g = l b_g \tag{17}$$

$$C_s = l B_s + C_s{}^0 \tag{18}$$

The proportionality coefficient is interpreted as an economic value of unit exergy. It is also termed less rigorously a unit exergy cost. Since exergy is a common measure of various types of energy, the coefficient l was assumed the same for both gas and solid. $C_s{}^0$ is economic value of solid in equilibrium with environment.

With the help of the relations, Equations (17) and (18), an expression for a thermodynamic form of the performance index can be obtained. Substituting Equations (17) and (18) into Equation (11) yields

$$K_e{}^p/l = K^p = \int_0^{\tau_k} (b_g - b_w)\,d\tau + B_{so} - B_{sk} \tag{19}$$

The quantity $K_e{}^p/l$ is designated as K^p, and it is termed as the process cost of production expressed in exergy terms or simply as the exergy cost of production. It is important to notice that the same expression can be obtained from the exergy balance as the total amount of exergy dissipated in the system taken with the minus sign and related to the mass unit of dry solid. Thus, exergy balance is an alternate method of obtaining the exergy performance index. The following important fact can be proven: Equations (18) and (19) give the same family of optimal trajectories and decisions even when coefficient l is different value for both phases.

The working form of the performance index expression should contain as its variables only the decision variables (i_g, X_g), the state variables (I_s, W_s), and the constant parameters (Le, N_g), or any known function of the above quantities. Therefore, all unit exergies in Equation (19) should be expressed as a function of the above variables. Substituting for this purpose Equations (10) and (11) into Equation (19), we obtain a common equation for both processes describing the performance index:

$$K^p = \int_{\tau_0}^{\tau_k} \left(b_g[i_g, X_g] - b_g \left\{ \frac{X_g + N_g X_s[I_s, W_s]}{1 + N_g}, \frac{Le\, i_g + N_g i_s[I_s, W_s] - N_g i_p (Le - 1)(X_g - X_s[I_s, W_s])(N_g + 1)^{-1}}{Le + N_g} \right\} \right) d\tau + B_s[I_{so}, W_{so}] - B_s[I_{sk}, W_{sk}] \quad (20)$$

Equation (20) describing the performance index as well as the state Equations (12) and (13), contain several implicit relationships. It should be pointed out that any implicit relationships in Equations (12), (13), and (20) (that is, functions $i_s[I_s, W_s]$, $X_s[I_s, W_s]$, $b_g[i_g, X_g]$, $B_s[I_s, W_s]$) are of a definite type: they are purely thermodynamic ones. Therefore, there is no difficulty in computing these functions when results of adsorption equilibrium are given. Since obtaining these functions is a purely thermodynamic task, we will not discuss this problem here; rather, the reader is referred to the Sieniutycz's work (1968, 1973) for details.

The present form of Equations (12), (13), and (20) is general in the sense that for this form it makes no difference which substances are used. In an optimization program, all of the discussed thermodynamic functions should be specified in analytic or tabular form.

EQUIVALENCE WITH MINIMUM REVERSIBILITY

Substituting Equation (15) and (16) into Equation (19), assuming isobaric process and applying the balance Equations (6) and (7) into the equation obtained yields

$$K^p = -T^o \left(\int_0^{\tau_k} (s_g - s_w) d\tau + S_{so} - S_{sk} \right) \quad (21)$$

The expression in the large parentheses represents total process irreversibility with the minus sign (total entropy production) as related to the unit mass of solid. Because the exergy cost K^p describes total exergy dissipation, the general form of the above equation is:

Total Exergy Dissipation =

$$- T^o \text{ (Total Entropy Production)} \quad (21a)$$

since the sign of the exergy dissipated is negative. The relationship between exergy dissipation and entropy production holds for every thermodynamic system. It is known as a Gouya-Stodola Law (Szargut and Petela, 1965). If the economic and exergy optimization gives the same results, then these results correspond with the minimum of the total entropy production (that is, minimum irreversibility). This fact justifies the use of the reversibility analysis for economic purposes which is very often done in literature (Bosniakowic, 1960). It should be noticed that the appropriateness of such an analysis can be proven exclusively on the basis of exergy (not entropy). This is so since only exergy (not entropy) can be related to the economic value of a substance.

SOME SIMPLIFICATIONS AND TRANSFORMATIONS OF THE PROCESS EQUATIONS

In accordance with the purpose of this article we will pay more attention to the very common case of an equilibrium fluidization $(N_g \to \infty)$ with the constant inlet gas humidity X_{go}. The respective equations are as follows:

$$\frac{dI_s}{d\tau} = i_g - i_s[I_s, W_s] \quad (22)$$

$$\frac{dW_s}{d\tau} = X_{go} - X_s[I_s, W_s] \quad (23)$$

$$\frac{dK^p}{d\tau} = b_g[i_g, X_{go}] - b_s[I_s, W_s] - \frac{dB_s}{d\tau} \quad (24)$$

Equation (24) was obtained by differentiating the integral describing the exergy costs, Equation (20), with respect to the variable upper limit of integration.

In order to exploit the fact that the process is autonomous it is advisable to transform Equations (22) through (24) into such a form in which time τ is a dependent variable. As a result, we will obtain an equivalent set of process equations which will not contain the dependent variable τ explicitly on the right-hand side. This dependent variable will allow the use of the Lagrangian multiplier to reduce the problem dimensionality. In order to obtain the set of equations having time as a dependent variable, we use the solid moisture content W_s as an independent variable. As a result, the set of Equations (22) through (24) takes the form

$$\frac{dI_s}{dW_s} = \frac{i_g - i_s[I_s, W_s]}{X_{go} - X_s[I_s, W_s]} \quad (25)$$

$$\frac{d\tau}{dW_s} = \frac{1}{X_{go} - X_s[I_s, W_s]} \quad (26)$$

$$\frac{dK^p}{dW_s} = \frac{b_g[i_g, X_{go}] - b_s[I_s, W_s]}{X_{go} - X_s[I_s, W_s]} - \frac{dB_s}{dW_s} \quad (27)$$

In the above formulation the state variables are solid enthalpy I_s and time τ; the decision is gas enthalpy i_g (X_{go} = const). The process parameters (N_g, Le) do not appear in the case of the equilibrium model.

THE CONTINUOUS FORM OF THE ORIGINAL* OPTIMIZATION PROBLEM

It is purposeful now to turn our attention to the formulation of our original optimization problem in the continuous version. We are investigating the minimum of the functional [obtained from Equation (27)]

$$K^p[i_g] = \int_{W_{so}}^{W_{sk}} \frac{(b_g[i_g, X_{go}] - b_s[I_s, W_s]) dW_s}{X_{go} - X_s[I_s, W_s]} + B_s[I_{so}, W_{so}] - B_s[I_{sk}, W_{sk}] \quad (28)$$

with the differential equations, Equations (25) and (26), and the additional conditions on temperature and concentration of the gas and material. The moisture content

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* We will distinguish the original problem (without the Lagrangian multiplier) from the transformed problem (when the Lagrangian multiplier is introduced).

W_s is always specified at the initial and final stages of the process, whereas the enthalpy I_s and the overall time τ may be prescribed or arbitrary. We recognize that this variational problem has the Bolza form.

THE LAGRANGIAN MULTIPLIER AND THE TRANSFORMED PROBLEM

Since thermodynamic relations in Equations (25) through (26) (that is, functions $i_s[I_s, W_s]$, $X_s[I_s, W_s]$ $b_s[I_s, W_s]$) are usually too complex in their analytical form, it is most appropriate to use dynamic programming as the method of optimization. Because of two restrictive equations, Equations (25) and (26), the dynamic programming recurrence equation is two-dimensional in the original version of our optimization problem. In order to simplify the computational procedure, the possibility of dimensionality reduction should be explored. Since the dependent variable τ is not explicitly present on the right-hand sides of Equations (25) through (27), there exists a possibility to use the Lagrangian multiplier and then transform the original problem to one with a single restrictive equation. This procedure leads to the problem of determining the functional extremum:

$$K^c = \int_{W_{so}}^{W_{sk}} \left(\frac{b_g[i_g, X_{go}] - b_s[I_s, W_s] + \lambda}{X_{go} - X_s[I_s, W_s]} \right) dW_s + B_s[I_{so}, W_{so}] - B_s[I_{sk}, W_{sk}] \quad (29)$$

with the differential restriction from Equation (25), appropriate initial and final conditions, as well as any additional restrictions imposed on the state and decision variables. It should be pointed out that final time in the transformed problem cannot be preassigned.

THE DISCRETE VERSION OF THE TRANSFORMED PROBLEM

In order to make use of computer facilities, Equation (25) and the performance index, Equation (29) should be written in the approximate difference form. The process index is

$$K^c[i_{g1} \ldots i_{gn}] = \sum_{1}^{N} \frac{(b_g[i_{gn}, X_{go}] - b_s[I_{sn}, W_{so} - n\Delta] + \lambda)\Delta}{X_s[I_{sn}, W_{so} - n\Delta] - X_{go}} + B_s[I_{sn-1}, W_{so} - (n-1)\Delta] - B_s[I_{sn}, W_{so} - n\Delta] \quad (30)$$

Then, we should find the extremum of the functional, Equation (30), with the difference restriction

$$I_{sn} - I_{sn-1} = \frac{(i_{gn} - i_s[I_{sn}, W_{so} - n\Delta])\Delta}{X_s[I_{sn}, W_{so} - n\Delta] - X_{go}} \quad (31)$$

and also the appropriate initial and final conditions as well as the additional restraints on the state and decision variables. (See following section.)

THE DYNAMIC PROGRAMMING ALGORITHM

With the help of Equations (30) and (31), and Bellman's (1957) Principle of Optimality, it is possible to derive a basic recurrence equation for the transformed problem. Designating

$$\min K^c[i_{g1} \ldots i_{gn}] = \tilde{F}_n^c[I_{sn}]$$

and taking advantage of the restrictive Equation (31) for the purpose of expressing the inlet gas enthalpy i_{gn} as a function of the material enthalpy before and after the stage (I_{sn-1} and I_{sn}, respectively), the following recurrence equation is obtained:

$$\tilde{F}_n^c[I_{sn}] = \min_{I_{sn-1}} \left\{ \frac{\left(b_g\left[i_s[I_{sn}, W_{so} - n\Delta] + \frac{(I_{sn} - I_{sn-1})(X_s[I_{sn}, W_{so} - n\Delta] - X_{go})}{\Delta}, X_{go} \right] - b_s[I_{sn}, W_{so} - n\Delta] + \lambda \right)\Delta}{X_s[I_{sn}, W_{so} - n\Delta] - X_{go}} + B_s[I_{sn-1}, W_{so} - (n-1)\Delta] - B_s[I_{sn}, W_{so} - n\Delta] + \tilde{F}_{n-1}^c[I_{sn-1}] \right\} \quad \text{with} \quad \tilde{F}_o^c = 0 \quad (32)$$

The above equation was solved for the constant inlet solid state $I_{so} = -1$ kcal/kg, $W_{so} = 0.1$ kg/kg ($t_{so} = 22.6°$C). The constant inlet gas humidity was accepted as that found in atmospheric air, $X_{go} = 0.008$ kg/kg. It was assumed that the maximum admissible inlet gas temperature $t_{g\max}$ is equal to 374°C.

The restriction on the solid parameters was assumed to have the form

$$X_s[I_{sn}, W_{sn}] \geqq X_{go} \quad (33)$$

This inequality establishes the boundary of solid state changes, since in the case of the drying process, the recurrence equation, Equation (32) must be solved within the boundary of the variables (I_s, W_s) where the evaporation direction is from solid to gas. This boundary should be known from drying equilibrium data. These data as well as thermodynamic functions of gas and solid were known from the work of Sieniutycz (1968, 1973). Since silicagel is a popular industrial adsorbent, the silicagel-water-air system was studied in the computations.

After minimization of the right-hand side of the recurrence Equation (32) and storage of optimal thermodynamic parameters of gas and solid before and after every state, the results of the optimal process time are computed. This is accomplished by means of the formula:

$$\hat{\tau}_n[I_{sn}] = \frac{\Delta}{X_s[I_{sn}, W_{so} - n\Delta] - X_{go}} + \hat{\tau}_{n-1}\{\check{I}_{sn-1}[I_{sn}]\} \quad (34)$$

which represents a difference form of Equation (26). The final process time, Equation (34) for $n = k$, depends upon the assumed value of the parameter λ as well as upon the state of the process (I_{sk}, W_{sk}). Iterating computations for the same boundaries of the state variables (I_{sk}, W_{sk}) and changing only the values of the parameter λ, one can obtain tables describing the relationship between the multiplier λ and the optimal process time $\hat{\tau}_k$. The example of this relationship is given in Table 1. The results pertain to the initial solid state $I_{so} = -1$ kcal/kg, $W_{so} = 0.1$ kg/kg ($t_{so} = 22.6°$C) and the final solid state $I_{sk} = 5$ kcal/kg, $W_{sk} = 0.06$ kg/kg ($t_{sk} = 40°$C). Using Table 1 we can find a numerical value of λ which solves the problem of an optimal transition of a solid from the constant initial state ($I_{so} = -1$ kcal/kg, $W_{so} = 0.1$

kg/kg) to the specified final state ($I_{sk} = 5$ kcal/kg, $W_{ks} = 0.06$ kg/kg) in a previously specified time τ_k. In order to obtain the optimal trajectories and decisions, we must once again perform the appropriate computations for that particular value of λ. This operation ends the computational procedure.

A complete flow diagram* of the program used in the computations of the optimal decisions and trajectories is described in the Sieniutycz's (1973) work, as well as a sample of the computational problem.

RESULTS OF THE COMPUTATIONS

The following values: the optimal costs $\tilde{K}_n{}^c[I_{sn}]$, the optimal solid enthalpy I_{sn-1}, the optimal gas enthalpy i_{gn} (or optimal gas temperature t_{gn}) and optimal time τ_n, are found as a result of our computations. The basic curves characterizing the optimal process are shown in Figure 2. Data on this graph pertains to the conditions pointed out in the preceding section. The trajectories, which are computed by the dynamic programming method, are designated with broken lines on the enthalpy-composition graph for silicagel. Also shown are the curves for the optimal costs, $\tilde{F}^c =$ const, and the optimal inlet gas temperatures, $t_g =$ const. One can notice that no specific curve exists for the maximum temperature $t_{g\max}$, but boundaries do exist for the solid-state variables (I_s, W_s). Inside of these boundaries the state change is connected with the constant gas temperature policy.

The graph of the optimal inlet gas temperature as a function of a dimensionless time τ is shown on Figure 3. Besides the various curves for $t_g[\tau]$, the graph on Figure 3 also contains the curves for constant final solid temperature t_{sk} and final solid moisture content W_{sk}. If on the discussed graph a point with the coordinates (t_{sk}, W_{sk}) is specified (characterizing the preassigned final material state) then the desired optimal control curve $t_g[\tau]$, and thus the process trajectory, can be found.

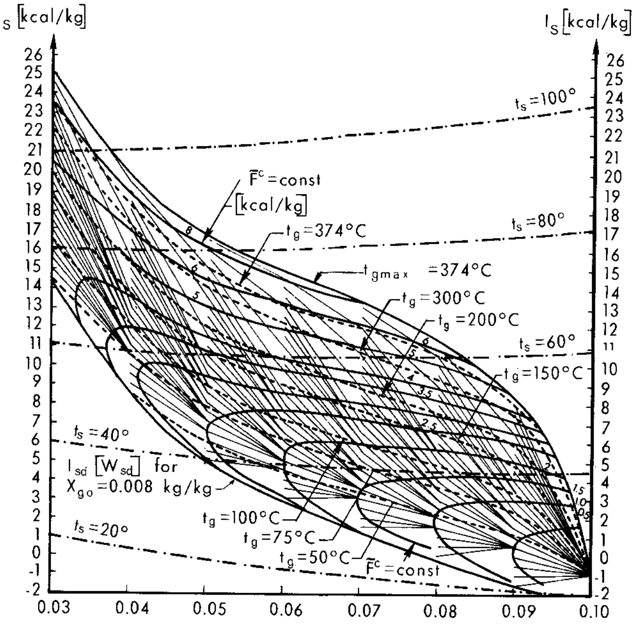

Fig. 2. Optimal trajectories, constant decision curves, and constant costs value curves on the enthalpy-moisture content diagram (for solid phase). The equilibrium problem ($N_g \rightarrow \infty$). Results for silicagel-water-air system, $t_{so} = 22.6°$C, $I_{so} = -1$ kcal/kg, $W_{so} = 0.1$ kg/kg, $\sigma = 0.2$ kcal/kg, $\Delta = 0.01$ kg/kg, $\lambda = 0.1$ kcal/kg, $X_{go} = 0.008$ kg/kg, $t_{g\max} = 374°$C.

Table 1. Relationship Between Multiplier λ and Optimal Process Time

τ_k	—	4	6	8	10	12	14	16	18	20
λ	$\frac{\text{kcal}}{\text{kg}}$	1.9	1.3	0.9	0.6	0.4	0.2	0.12	0.08	0.06

The family of curves describing the function $t_g[\tau]$ on Figure 3, indicates that in order to attain any final state of material the optimal gas inlet temperature increases monotonically in time until the maximum admissible gas temperature $t_{g\max}$ is achieved. After this, the optimal policy remains constant in time (the optimal gas temperature t_g is equal to the maximum temperature $t_{g\max}$). When there are no restrictions on the gas temperature the difference between the temperatures of both phases, that is, the process driving force, remains nearly equal in time.

The common method of conducting the fluidization processes is by the use of the constant inlet gas temperature policy. The results of optimization show that the optimum of costs corresponds in general with the increasing gas temperature policy and that the constant inlet gas temperature policy can be only a final part of the optimal policy when an upper limit of the gas temperature is assumed. The optimal inlet gas temperature policy can be simply adjusted in the batch fluidization case when the inlet gas stream is heated with the intensity which is variable in time (variable flow of heating medium or variable intensity of electric heating with the use of an autotransformer). In the horizontal exchanger case, the optimal policy can be realized only approximately when inlet gas is divided into several streams flowing to different sections of the apparatus and when each gas stream is heated to a definite temperature.

There are two basic aims behind the optimization of any physical process. The optimal results can be used either to design a new and better process already taking place in a given apparatus, or they can be used in the design of new equipment. Because the cost of production was used as a performance criterion, the results obtained can be quantitatively* used only for the first case. In this case, they should be used for the design of the fluidization process with variable inlet gas conditions. The optimization solution gives a controlling algorithm for the definite process time τ_k. The example below should explain the procedure of process design.

EXAMPLE

The silicagel is dried by heated atmospheric air ($X_{go} = 0.008$ kg/kg) in the batch fluidization process. The initial state of the solid is $t_{so} = 22.6°$C, $W_{so} = 0.1$ kg/kg ($I_{so} = -1$ kcal/kg). The required final solid state is $t_{sk} = 40°$C, $W_{sk} = 0.06$ kg/kg ($I_{sk} = 5$ kcal/kg). The total process time is $\eta_k = 25$ min. The superficial gas mass velocity during fluidization is $G/A = 0.5$ kg/m² s, the amount of solid per unit area of apparatus cross section is equal to

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* The flow diagram has been deposited as Document No. 01995 with National Auxiliary Publications Service (NAPS), c/o Microfiche Publications, 305 E. 47th Street, N.Y., N.Y. 10017 and may be obtained for $2.00 for microfiche and $5.00 for photocopies.

* However, qualitative results obtained here can be applied also for the second case. These are pointed out in the last section of this article.

** In order not to involve additional graphical and tabular data the initial data in this example were assumed in a suitable manner. They correspond with the data in Figures 2 and 3. This does not have influence on the method of solution in our example.

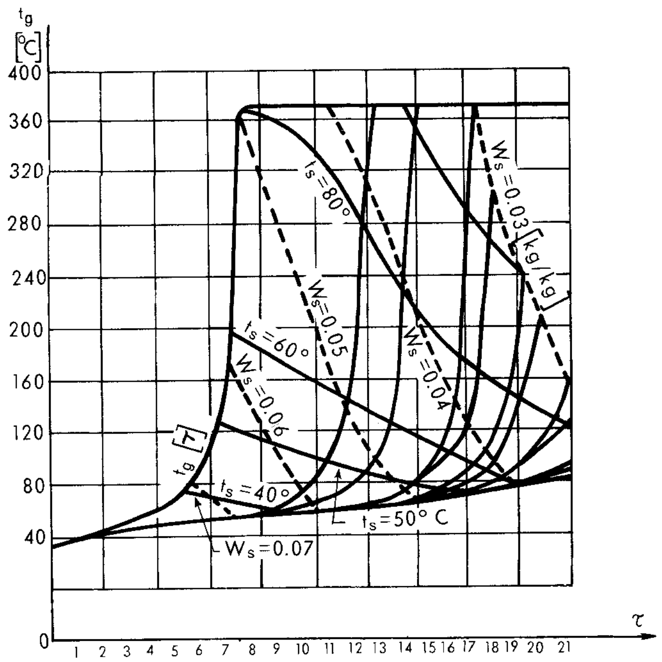

Fig. 3. Optimal drying gas temperatures versus the dimensionless time τ. Lines of constant temperatures t_s and moisture contents W_s of solid on t_g — τ plane. Input data as on Figure 2.

S_o/A = 45 kg/m^2. Our task is to find the optimal trajectory and optimal decisions corresponding with the minimum costs of production.

Solution

The optimization results contain data for process final time which allow us to obtain a table λ — τ_k for t_{sk} = 40°C, W_{sk} = 0.06 kg/kg (I_{sk} = 5 kcal/kg) (Table 1).

The dimensionless time of the process is

$$\tau_k = \frac{\eta_k G/A}{S_o/A} = \frac{25 \cdot 0.5 \cdot 60}{45} = 16.7$$

The Lagrangian multiplier value corresponding to a given solid final state can be obtained from Table 1. For τ_k = 16.7, we obtain $\lambda \simeq 0.1$ kcal/kg. This is a particular value of λ for which the optimal trajectory and optimal decisions curves should be found. The optimal results for this value of λ are presented in Figures 2 and 3. Our solution corresponds with that trajectory and decision curve which has the final state t_{sk} = 40°C, W_{sk} = 0.1 kg/kg. Since in this case the accuracy of graphical results is poor, the tables containing the dynamic programming results have been used to obtain more exact results. These results are given in Table 2.

THE LAGRANGIAN MULTIPLIER PROPERTIES AND SOME QUALITATIVE RESULTS

The form of expression for the performance index K^c, Equation (29), indicates that the quantity K^c plays the role of total process cost, which is the sum of the production and investment costs. Consequently, the Lagrangian multiplier can be interpreted as the unit apparatus price expressed in the appropriate units. There is a growing interest in the investigation of how numerical values of λ are related to typical fluidized drying apparatus. With such information, the optimization solution will give information about the optimal process time directly. At the present time, however, it is possible to obtain several important qualitative conclusions when the apparatus price λ is considered as a variable parameter. Such consideration deals with various optimal processes having the same initial and final states which take place in different apparata (different values of λ). The optimal time of such processes increases when λ decreases (see Table 1). In the limit when λ approaches zero (very inexpensive apparatus) the optimal time τ_k approaches infinity. Corresponding to this case there is a limiting numerical value of optimal costs $\tilde{K}^p$ and $\tilde{K}^c$ equal to zero. This value of costs is the result of conducting the process in which the substance final state (I_{sk}, W_{sk}) is attained from the initial state (I_{so}, W_{so}) without the existence of driving forces between both phases ($i_g = i_s$, $X_g = X_s$). Such a process has a rate approaching zero and is known in classical thermodynamics as quasistatic. Thus, a quasistatic process can be optimal only in the case when the apparatus price approaches zero.

In the case when the apparatus price is greater than zero ($\lambda > 0$), the optimal final time has a definite value. The driving forces $i_g - i_s$ and $X_g - X_s$ are then different from zero. In this case, only the irreversible process will be optimal. The relation between apparatus price, process intensity, and process time has not been sufficiently examined so far in engineering literature.

For very expensive apparatus, only very intensive processes can be optimal; they are characterized by a large irreversibility index (that is, large entropy production or large exergy dissipation). This means that the design of expensive apparatus must be associated with the design of intensive optimal processes to assure short process time. This result should be helpful in avoiding basic mistakes in the design of new equipment.

TABLE 2. RESULTS OF OPTIMAL QUANTITIES ALONG THE PROCESS OPTIMAL TRAJECTORY

W_s	kg/kg	0.10	0.09	0.08	0.07	0.06
t_s	°C	22.6	28.0	32	36	40
t_g	°C	33.1	45.2	49.5	50.5	55
τ	—	0	3.5	5.0	6.5	16.7
η	min	0	5.25	7.5	9.75	25

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NOTATION

A = apparatus cross-sectional area, m^2
a = solid specific surface, m^2/m^3
B_s = solid exergy per mass unit of absolute dry substance, kcal/kg, kJ/kg
B_{sn} = solid exergy after stage n, kcal/kg, kJ/kg
B_{so} = initial solid exergy, kcal/kg, kJ/kg
B_{sk} = final solid exergy, kcal/kg, kJ/kg
b_g = inlet gas exergy, kcal/kg, kJ/kg
b_s = exergy of gas in equilibrium with solid, kcal/kg, kJ/kg
b_w = outlet gas exergy, kcal/kg, kJ/kg
C_s = solid economic value per mass unit of dry solid, \$/kg
C_{so}, C_{sk} = initial and final solid unit economic values, respectively, \$/kg
$\overline{C_g}$ = heat capacity of gas per mass unit of dry gas, kcal/kg °K, kJ/kg °K

c_g, c_w = gas economic value per mass unit of dry gas for inlet and outlet gas, respectively, \$/kg

$\tilde{F}_n^c[I_s]$ = optimal overall exergy cost as a function of vari-variables I_s, n, λ

G = gas flow rate (see Figure 1, schemes A and B), kg/h, kg/s

$HTU_g = G/K_g'A$ = height of the mass transfer unit, m

h = height of fluidized layer, m

$I_{\max}$ = maximum solid enthalpy corresponding to the gas enthalpy $i_{g\max}$ or temperature $t_{g\max}$, kcal/kg, kJ/kg

I_s = solid enthalpy per unit mass of absolute dry substance, kcal/kg, kJ/kg

I_{sn} = enthalpy of solid from stage n, kcal/kg, kJ/kg

I_s^o = enthalpy of solid in equilibrium with environment, kcal/kg, kJ/kg

I_{sd} = low restrictions line enthalpy (Figure 2), kcal/kg, kJ/kg

I_{sk} = final solid enthalpy, kcal/kg, kJ/kg

i_g = gas enthalpy per unit mass of absolute dry gas, kcal/kg, kJ/kg

also

i_g = inlet gas enthalpy, kcal/kg, kJ/kg

i_p = partial enthalpy of moisture in vapor state (assumed as constant value), kcal/kg, kJ/kg

i_s = enthalpy of gas in equilibrium with solid, kcal/kg, kJ/kg

i_w = gas outlet enthalpy, kcal/kg, kJ/kg

i_g^o = enthalpy of gas in equilibrium with environment, kcal/kg, kJ/kg

K^p = cost of production expressed in exergy terms, kcal/kg, kJ/kg

K_e^p = cost of production expressed in economic terms, \$/kg

$\tilde{K}_n^c[I_s]$ = overall optimal exergy cost as a function of variables, I_s, n and τ, kcal/kg, kJ/kg

K_g' = modified overall mass transfer coefficient, kg/m²h

k = overall number of stages corresponding to the final state

$Le = K_g'\overline{C}_g/\alpha$ = Lewis factor

l = economic value of unit exergy, \$/kcal

$N_g \equiv \dfrac{h}{HTU_g}$ = number of mass transfer units

n = current stage number

P^o = pressure in environment, ata

P_s, P_g = pressure in solid phase and gas phase, respectively, ata

S = dry solid flow rate, kg/h

S_o = dry solid mass in batch fluidization process, kg

S_s = solid entropy per unit mass of dry solid, kcal/kg °K, kJ/kg °K

S_s^o = entropy of solid with equilibrium with environment, kcal/kg °K, kJ/kg °K

S_{so}, S_{sk} = initial and final entropy of solid, respectively, kcal/kg °K, kJ/kg °K

s_g = gas entropy per unit mass of dry gas, kcal/kg °K, kJ/kg °K

s_g^o = entropy of solid with equilibrium with environment, kcal/kg °K, kJ/kg °K

s_w = entropy of outlet gas, kcal/kg °K, kJ/kg °K

T^o = absolute temperature of environment, °K

t_g = inlet gas temperature, °C

$t_{g\max}$ = maximum inlet gas temperature, °C

t_s = solid temperature, °C

t_{so} = initial solid temperature, °C

V = fluidized layer volume, m³

V_s^o = volume of solid in equilibrium with environment (per mass unit of dry solid), m³/kg

v_g^o = volume of gas in equilibrium with environment (per mass unit of dry gas), m³/kg

W_s = absolute moisture content of solid (as mass of moisture per mass unit of dry solid), g/kg, kg/kg

W_{sn} = moisture content of solid from stage n, g/kg, kg/kg

W_{sd} = low restrictions line moisture content of solid (Figure 2), g/kg, kg/kg

W_{so} = initial solid moisture content, g/kg, kg/kg

W_{sk} = final solid moisture content, g/kg, kg/kg

W_s^o = moisture content of solid in equilibrium with environment, g/kg, kg/kg

X_g = humidity of gas in equilibrium with environment, g/kg, kg/kg

Greek Letters

α = overall heat transfer coefficient, kcal/m² °C h, kg/m² °C h

$\Delta \equiv W_{sn} - W_{sn-1}$ solid moisture content increment, g/kg, kg/kg

η = residence time (batch fluidization), min, h

μ^o = chemical potential of moisture in environment, kcal/kg, kJ/kg

σ = solid enthalpy increment, kcal/kg, kJ/kg

$$\tau \equiv \begin{cases} G/S \\ \text{horizontal fluidized exchanger} \\ G\eta/S_o \\ \text{batch fluidization} \end{cases} \quad \begin{matrix}\text{dimensionless} \\ \text{time of the} \\ \text{process}\end{matrix}$$

τ_k = dimensionless final time

Subscripts

e = economic value

g = gas, gas inlet

k = final state

n = stage n

o = initial state

p = vapor

s = solid

w = outlet

Superscripts

c = overall cost

o = equilibrium with environment

p = production cost

$\sim$ = functional optimum

$\wedge$ = optimal final quantity

$\smile$ = optimal quantity before stage

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